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Particle-sampling statistics in laser anemometers: sample-and-hold systems and saturable systems

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The output statistics of a laser anemometer operating in a low particle density are discussed. A rigorous derivation is given for the influence of two popular data-handling algorithms on these statistics. In particular it is shown that the *measured* statistics can differ from those of the flow statistics and from the particle-arrival statistics. The variables that control the statistical regime are derived and quantitative estimates are given for their ranges of influence.

The first system discussed is a sample-and-hold system where the output is a piecewise-continuous signal obtained by holding the last processor measurement until a new one is obtained. The second system is one where an attempt is made to store all the measurements for processing, but which contains a rate-limiting device. Because of this device, some measurements may be lost when the particle rate is high. This system is referred to as a saturable system.

In both cases it is found that the statistics of the output depend on the product of the mean particle rate and the flow correlation time as well as the flow statistics. The statistics of the saturable system also depend on the ratio of the mean particle rate to the maximum rate at which measurements can be accepted by the system. Because of this, the statistics of both systems depend on the particle density.

Attainable conditions are demonstrated, where the output velocity measurement statistics are essentially identical with the flow statistics.

1. Introduction

A laser anemometer detects the scattered light from macroscopic particles suspended in a flow as they pass through a probe volume. The processing schemes used for the scattered signals can be divided in two groups, namely single-particle velocity processors and many-particle velocity processors. The last group of processors deal with continuous signals and will work only if the number of particles in the measuring volume is large. Under certain conditions they can give the instantaneous velocity of the flow, from which mean values and other statistics can be deduced (George & Lumley 1973; Lading & Edwards 1975). An ideal single-particle velocity processor can detect the velocity of every particle passing the probe volume, whatever the particle density. These processors should be much more flexible than the many-particle velocity processor. The data obtained by the single-particle processor are random samples of the flow velocity. The sample rate of a given velocity will, in general, depend on the velocity.

In an ideal situation, the particle flow rate is proportional to the speed. In a paper of McLaughlin & Tiedermann (1973) it was pointed out that this would give a bias in the measured mean velocity if a simple arithmetic average was performed. In a later paper by Giel & Barnett (1979) the existence of this bias was questioned, based mainly on a lack of experimental verification. In other reports, such as that of Stevenson *et al.* (1980), the bias was experimentally verified but seemed to depend on the particle concentration and vanished in the limit of high particle densities. The last effect will be shown to be due to the fact that the data-handling system was saturated at high particle rates; a case described qualitatively by Edwards (1979, 1981) and Giel & Barnett (1979). Similar 'saturation' results were obtained by Erdmann & Tropea (1981) in their detailed computations on 'controlled' processors.

This paper is, in part, an attempt to reconcile some of the confusion existing about the measurement of the particle statistics. In it, we show how the data-handling system can strongly influence the results of an experiment. For the two systems examined, we derive the experimentally accessible parameters that control the statistics of the output, and the asymptotic limits of behaviour of the statistics and estimates of the range of parameter values where the asymptotic behaviour is observed.

In an actual experiment, the mean sampling rate of each velocity is a complicated function of the system parameters. For instance, the geometry, seeding method, and processor electronics can all affect the sampling rate. Here, we formally divide the laser anemometer system into two parts:

(i) *The initial processor.* This is the device that makes the individual measurements. Usually this is a 'counter'. Its settings, along with the geometry, determines the *input* particle statistics. In this paper, the input processor is assumed to have no measurement error and no reset (dead) time. For clarity of presentation, it will initially be assumed that the mean output rate from the initial process, for any velocity, is the particle density times the measurable volume swept through the measurement region per unit time by that velocity. This restriction will be relaxed in §3.

(ii) *Data-handling processor.* This is the part of the system that operates on the initial processor data to produce a system *output*. Two types of data-handling schemes are examined. The first is the sample-and-hold processor – a pseudo-analog device wherein the result of the last successful measurement is held until the next successful measurement. The output of the sample-and-hold is time-averaged. The second processor examined is a data-handling system with a 'dead' time. This will be denoted a 'saturable' system. A counter with a significant reset time or a data-storage buffer with a fine acquisition rate are typical examples of such a system. Note that a real counter is modelled as an ideal counter with no dead time followed by a dead time.

Both kinds of data-handling system are in use. However, their effect on the output statistics was heretofore not well understood, although the qualitative behaviour at some asymptotes has been discussed in the literature. Here we derive a rigorous framework within which the problems can be examined. In both cases, the form of the velocity probability density for the output will be derived. These will result from the computation of conditional probabilities, since the probability of seeing a given velocity measurement in the output, at a given time, is a function of the history of the system. All the mean moments of the output velocity can be computed from this probability density.

For the two systems described here, at the extreme of very low particle density, the output statistics will be shown to tend to those of the 'individual-realization' system described first by McLaughlin & Tiedermann (1973) and extended to three dimensions by Buchhave (1976). At the other extreme, high particle density, the statistics will be shown to tend toward an 'unbiased' system. More precise definitions of these terms will be given below.

2. The sample-and-hold processor

The output of a sample-and-hold processor can be described in mathematical terms by

$$\hat{v}(t) = \sum_i v_i \xi_i(t), \tag{1}$$

where $\hat{v}(t)$ is the output velocity at the time t, and v_i is the velocity at the time t_i when the last particle was measured. $\xi_i(t)$ is given by

$$\xi_i(t) = \begin{cases} 1 & (t_i \leqslant t < t_{i+l}), \\ 0 & \text{otherwise}. \end{cases} \tag{2}$$

All velocities are considered to be vectors unless otherwise noted.

The probability density function for $\hat{v}(t)$, here denoted $g_{\text{SH}}(\hat{v})$, can be found by summing the time average of all the ξ_i which have the velocity $\hat{v}$. This gives formally

$$\begin{aligned} g_{\text{SH}}(\hat{v}) &= \sum_{v_i=\hat{v}} \frac{1}{T_\text{a}} \int_t^{t+T_\text{a}} \xi_i(t')\,\text{d}t' \\ &= \frac{1}{T_\text{A}} \sum_{v_i=\hat{v}} (t_{i+1}-t_i). \end{aligned} \tag{3}$$

By averaging (3) over all times of arrivals and over all velocities different from $\hat{v}$, (3) becomes

$$\langle g_{\text{SH}}(\hat{v})\rangle\,\Delta\hat{v} = p_1(\hat{v})\,\Delta\hat{v} \int_0^\infty \tau p(\tau|\hat{v})\,\text{d}\tau, \tag{4}$$

where $p_1(\hat{v})\,\Delta\hat{v}$ is the probability per unit time for the arrival of a particle with the velocity $\hat{v}$, and $p(\tau|\hat{v})$ is the conditional probability density for the time τ between two following particles given the velocity $\hat{v}$ of the first one. $\langle g_{\text{SH}}(\hat{v})\rangle$ is the probability of the output of the sample-and-hold detector being $\hat{v}$.

The expected value of any function of $\hat{v}$, say $F(\hat{v})$, can then be found from (4):

$$\langle F(\hat{v})\rangle = \sum_{\hat{v}_i} \langle g(\hat{v}_i)\rangle\, F(\hat{v}_i)\,\Delta\hat{v}_i, \tag{5}$$

where the summation is taken over all possible velocities.

In order to calculate (4), the statistics of the particles must be known. Since the positions of the particles are independent and uniformly distributed, the probability for the arrival of two particles at the times t_1 and t_2 with the velocities v_1 and v_2 will be conditionally independent, i.e.

$$p_\text{A}(t_1, t_2, v_1, v_2) = p_\text{A}(t_1, v_1)\, p_\text{A}(t_2, v_2) \tag{6}$$

where $p_\text{A}(t_1, v_1)$ is the conditional probability for the arrival of a particle in the time interval $(t_1, t_1+\Delta t)$ with the velocity v_1 given by

$$p_\text{A}(t_1, v_1) = \frac{1}{\Omega}\alpha(v_1); \tag{7}$$

$\alpha(v)$ is the *measurable* volume of fluid swept through the measurement volume per unit time if the velocity is v. In general, α is a very complicated function of the velocity (see Buchhave 1975). It can depend on the flow angle and on the magnitude of the velocity in a nonlinear fashion. The quantity α is a function of time through the velocity $v(t)$. We will occasionally invoke this implicit dependence by writing $\alpha(t)$. If α is written with no argument it should be understood to be $\alpha(v)$.

Note that α is positive-definite. Thus a flow with a zero mean velocity will not have a zero mean value of α.

Ω is the volume of our world (box normalization). The number of particles in the box is denoted by M, and it is assumed that they all arrive within the time interval $(0, T_a)$. The relations between M, Ω and T_a are

$$\rho = \frac{M}{\Omega}, \quad \frac{1}{\Omega}\langle\alpha\rangle T_a = 1, \tag{8}$$

where ρ is the particle density. As usual Ω, M and T_a will not appear in the expression for the expected values of physical quantities.

2.1. *Calculation of p_1*

The probability per unit time for the arrival of a particle with velocity v, $p_1(v)\,\Delta v$, is proportional to the product of the probability $p(v)$, for a velocity v existing, and the probability $\alpha(v)/\langle\alpha\rangle$ of *measuring* v if the velocity is v.

2.2. *Calculation of $p(\tau|v)$*

From the condition that gave (6) and from (7) it can be deduced that the probability for the arrival of n particles in a time interval (t_1, t_2), given the velocity in the same time interval, follows a Poisson distribution:

$$p(n, t_1, t_2) = \frac{(N(t_2-t_1))^n}{n!}\mathrm{e}^{-N(t_2-t_1)}, \tag{9}$$

where

$$N(t_2-t_1) = \rho\int_{t_1}^{t_2}\alpha(t')\,\mathrm{d}t'. \tag{10}$$

The expression $N(t_2-t_1)$ is the expected number of particles swept through the measurement volume between t_1 and t_2. The conditional probability $p(\tau|v)$ for the time τ between two following particles, given the velocity of the first particle, is equal to the probability of no particle arrival in the time interval $(0,\tau)$, *and* the arrival of a particle at the time τ given the velocity at 0 was v. This gives (up to a normalization constant)

$$p(\tau|v)\,\Delta\tau = \langle\rho\alpha(\tau)\,\mathrm{e}^{-N(\tau)}|v\rangle\,\Delta\tau. \tag{11}$$

Now that we have an expression for $p(\tau|v)$, $\langle g_{\mathrm{SH}}(\hat{v})\rangle$ can be computed from (4) as

$$\langle g_{\mathrm{SH}}(\hat{v})\rangle = \frac{\alpha(\hat{v})\,p(\hat{v})}{\langle\alpha\rangle\,\Omega_g}\int_0^\infty \tau\,\langle\rho\alpha(\tau)\,\mathrm{e}^{-N(\tau)}|\hat{v}\rangle\,\mathrm{d}\tau. \tag{12}$$

The normalization constant Ω_g can be shown to be $1/\rho\langle\alpha\rangle$, thus $\langle g_{\mathrm{SH}}(\hat{v})\rangle$ is given by

$$\langle g_{\mathrm{SH}}(\hat{v})\rangle = \rho\alpha(\hat{v})\,p(\hat{v})\int_0^\infty \tau\langle\rho\alpha\,\mathrm{e}^{-N(\tau)}|\hat{v}\rangle\,\mathrm{d}\tau. \tag{12a}$$

Integrating (12*a*) by parts (noting that $\rho\alpha\,d\tau$ is dN), an alternative form is found:

$$\langle g_{SH}(\hat{v})\rangle = \rho\alpha(\hat{v})\,p(\hat{v})\int_0^\infty \langle e^{-N(\tau)}|\hat{v}\rangle\,d\tau. \tag{13}$$

The integrand is the probability of there being no measurement in the time τ if the initial velocity was $\hat{v}$. The expected value in the integrand can thus be computed by knowing the probability for the expected number of particles $N(\tau)$ being swept through the measurement volume in time τ, *if* the initial velocity was $\hat{v}$:

$$\langle e^{-N(\tau)}|\hat{v}\rangle = \int_0^\infty e^{-N}p(N|\tau,\hat{v})\,dN. \tag{14}$$

The probability of seeing $\hat{v}$ in the output is seen to be the product of the probability of there being a measured $\hat{v}$ and the probability of the persistence of that value in the output. The persistence of a value in the output is a function of the number of particles swept through the volume in the measurement time. In a stochastic flow, the number swept through in that time is conditionally dependent on the initial value of the velocity.

In order to compute exactly the probability density function $\langle g_{SH}(\hat{v})\rangle$ for the output, the flow statistics must be known, at least, to second order in time. These statistics are rarely known, especially before a measurement is made.

Define τ_c, the velocity persistence time for the flow. It is the microscale for the turbulence. During this interval the velocity essentially stays $v(0)$.

If $\rho\alpha(v)\tau_c \gg 1$, the number of particles swept through the volume during the persistence time is large, and the integral in (13) can be approximated by $1/\rho\alpha(v)$, since the integrand will be essentially zero for $\tau > \tau_c$. In this limit

$$\langle g_{SH}(\hat{v})\rangle = p(\hat{v}). \tag{15}$$

The probability of observing velocity $\hat{v}$ in the output is identical with the probability of there being a velocity $\hat{v}$ in the flow. The output is expected to be able to change faster than the flow. This is the unbiased limit.

If $\rho\alpha(v)\tau_c \ll 1$ (the number of particle swept through in the persistence time is small) then the integrand can be approximated by

$$\langle e^{-N}|\hat{v}\rangle \approx e^{-\langle N\rangle} = e^{-\rho\langle\alpha\rangle\tau}.$$

In this limit, the time interval between particles is so large that there is no correlation between the velocity at the end of the interval and the initial measured velocity. For such large times, the expected volume swept out is proportional to the mean velocity $\langle v\rangle$. The probability density for the measured output will be given by

$$\frac{\alpha(\hat{v})}{\langle\alpha\rangle}p(\hat{v}). \tag{16}$$

The statistics for the sample-and-hold processor become the same as those of an 'individual-realization' processor in the limit of a low measurement rate.

High and low measurement rate is now understood to mean that the characteristic measurement time $\tau_M = 1/\rho\langle\alpha\rangle$ is small or large compared with the flow persistence time τ_c. In order to give some guidelines to the experimenter, we give below an approximate form for $\langle g_{SH}(\hat{v})\rangle$.

The integral in the expression (13) for $\langle g_{SH}(\hat{v})\rangle$ is seen to be evaluated by a double integral (see (14)). Such a computation should not be sensitive to the precise form of $p(N|\tau, v)$.

The assumed approximate behaviour of $\langle N(\tau)|v\rangle$ is as follows:

$$\langle N(\tau)|v\rangle = \begin{cases} \rho\alpha(0)\tau & (\tau < \tau_c), \\ \rho\langle\alpha\rangle(\tau-\tau_c)+\rho\alpha(0)\tau_c & (\tau \geqslant \tau_c). \end{cases} \tag{17}$$

The velocity is assumed to stay v up to the persistence time τ_c, and then to change abruptly to the expected velocity $\langle v\rangle$. Any real turbulence flow will have the same asymptotes, but will change smoothly from one limit to the other around the time τ_c.

For $\tau < \tau_c$ we shall assume that

$$p(N|\tau, v) = \delta(N - \langle N(\tau)|v\rangle),$$

and for $\tau \geqslant \tau_c$

$$p(N|\tau, v) = \frac{1}{(2\pi\sigma_N^2)^{\frac{1}{2}}}\exp\left[-\frac{(N-\langle N(\tau)|v\rangle)^2}{2\sigma_N^2}\right],$$

where σ_N^2 is the variance $\langle(N-\langle N(\tau)|v\rangle)^2\rangle$. The variance σ_N^2 is approximated by an eddy-diffusivity formula (Hinze 1959), i.e.

$$\sigma_N^2 = \langle(N-\langle N(\tau)|v\rangle)^2\rangle, \tag{18a}$$

$$\sigma_N^2 = 2\rho^2\sigma_\alpha^2\tau_c(\tau-\tau_c), \tag{18b}$$

where σ_α^2 is the variance of α.

Using the above, $\langle g_{SH}(\hat{v})\rangle$ is approximately given by

$$\langle g_{SH}(\hat{v})\rangle \approx p(\hat{v})\left\{1-\left[1-\frac{\alpha(\hat{v})}{\langle\alpha\rangle}\left(1+\frac{\tau_c}{\tau_M}\frac{\sigma_\alpha^2}{\langle\alpha\rangle^2}\right)\right]\exp\left[-\frac{\tau_c}{\tau_M}\frac{\alpha(\hat{v})}{\langle\alpha\rangle}\right]\right\}. \tag{19}$$

If the temporal statistics of the flow are jointly Gaussian, $\langle g_{SH}(\hat{v})\rangle$ can be computed numerically. The details of this computation are given in the appendix. Values of $\langle g_{SH}(\hat{v})\rangle/p(\hat{v})$ for the approximate formula (19) and the exact Gaussian computation are given in table 1. Note that the approximate formula closely follows the exact formula, differing at most by 6%.

It is doubtful that differences between the approximate formula and the exact formula could be measured in an actual experiment especially if the mean or the variance of the velocity was measured. An estimate of the regimes of asymptotic behaviour can now be made:

$$\rho\langle\alpha\rangle\tau_c = \frac{\tau_c}{\tau_M} > 10. \tag{20}$$

In this regime the statistics are essentially unbiased:

$$\rho\langle\alpha\rangle\tau_c = \frac{\tau_c}{\tau_M} < 0.1. \tag{21}$$

In this regime, the statistics are those of an 'individual-realization' system.

If only one component of the velocity is measured, the probability density for that component can be derived from $\langle g_{SH}(\hat{v})\rangle$ by integrating over the other coordinates as follows:

$$\langle g_{SH}(\hat{v}_x)\rangle = \iint\langle g_{SH}(\hat{v})\rangle\,d\hat{v}_y\,d\hat{v}_z, \tag{22}$$

	$\frac{\alpha}{\langle\alpha\rangle} = 1.5,\ \frac{\sigma_\alpha}{\langle\alpha\rangle} = 0.2$	
τ_M/τ_c	Equation (19)	Gaussian
0.01	1.000	1.000
0.1	1.000	1.000
0.2	1.000	1.004
0.5	1.031	1.032
1.0	1.125	1.091
2.0	1.251	1.182
5.0	1.379	1.309
10.0	1.436	1.382
100.0	1.493	1.483
	$\frac{\alpha}{\langle\alpha\rangle} = 0.5\ \ \frac{\sigma_\alpha}{\langle\alpha\rangle} = 0.2$	
τ_M/τ_c	Equation (19)	Gaussian
0.01	1.000	1.000
0.1	0.999	0.9883
0.2	0.969	0.906
0.5	0.832	0.808
1.0	0.709	0.725
2.0	0.619	0.650
5.0	0.551	0.579
10.0	0.526	0.546
100.0	0.503	0.506

TABLE 1

$$\langle g_{\mathrm{SH}}(\hat{v}_x)\rangle = \rho\alpha(\hat{v}_x)p(\hat{v}_x)\int_0^\infty \langle \mathrm{e}^{-N(\tau)}|\hat{v}_x\rangle_{yz}\,\mathrm{d}\tau. \tag{23}$$

The form of the probability density for one component has the same form as that for the entire velocity vector. The expected value in the integrand has the same meaning as before except that it is now averaged over all y- and z-velocities. The limits for the asymptotic behaviours remain the same.

2.3. *The saturable system*

The saturated detector can be described as a detector with a dead time T, i.e. if the detector measures the velocity of a particle, at $t = 0$, it is first able to measure *another* particle after a time interval T has elapsed. If the time of arrival between the particles is much smaller than the dead time T, the flow velocity at the output will be sampled at a rate which is essentially independent of the particle arrival rate. In order to describe the behaviour of the saturable system, the statistics of the measurement have to be found as in the case of the sample-and-hold processor.

First, the probability for the acceptance of a particle in the output given its velocity will be calculated.

The conditional probability $p_{\mathrm{m}}(0, v, T)$ that a particle is accepted at the time $t = 0$, given the velocity v, is the probability that a particle arrives at $t = 0$ and that no particles have been accepted in the time interval T before $t = 0$, i.e.

$$p_{\mathrm{m}}(0, v, T)\,\Delta t = \rho\alpha(0)\,\Delta t\left(1 - \int_{-T}^{0} p_{\mathrm{m}}(t, v, T)\,\mathrm{d}t\right). \tag{24}$$

A solution to this integral equation has been found to be (up to a constant)

$$p_{\mathrm{m}}(t, v, T) = \frac{\rho\langle\alpha(t)\,|\,v\rangle}{1+\rho\int_{-T}^{0}\langle\alpha(t')\,|\,v\rangle\,\mathrm{d}t'}, \tag{25}$$

where $\langle\alpha(t)\,|\,v\rangle$ is the expected volume swept out in time t *if* the velocity at $t = 0$ is v.

For $t = 0$ this yields

$$p_{\mathrm{m}}(0, v, T) = \frac{\rho\alpha(v)}{1+\rho\int_{-T}^{0}\langle\alpha(t')\,|\,v\rangle\,\mathrm{d}t'}, \tag{26}$$

The function $p_{\mathrm{m}}(0, v, T)$ is the expected number of measurements per second, in the output, if the velocity is v. To compute $p_{\mathrm{m}}(0, v, T)$ one needs to compute the expected number of particles swept out in the previous time T, if the velocity is now v. By normalizing the function $p_{\mathrm{m}}(0, v, T)$, the statistics for the saturated detector are obtained:

$$\begin{aligned}\langle g_{\mathrm{SD}}\hat{v})\,|\,T\rangle &= \frac{p_{\mathrm{m}}(0,\hat{v},T)}{\langle p_{\mathrm{m}}(0,\hat{v},T)\rangle}p(\hat{v}) \\ &= \frac{\alpha(\hat{v})\,p(\hat{v})}{1+\rho\int_{-T}^{0}\langle\alpha(t')\,|\,\hat{v}\rangle\,\mathrm{d}t'}\left\langle\frac{\alpha(\hat{v})}{1+\rho\int_{-T}^{0}\langle\alpha(t')\,|\,\hat{v}\rangle\,\mathrm{d}t'}\right\rangle^{-1},\end{aligned} \tag{27}$$

where the average is taken over $\hat{v}$.

Again, if the statistics of α are known to second order in time, the function $\langle g_{\mathrm{SD}}(\hat{v})\,|\,T\rangle$ can be computed. In order to examine the behaviour of $\langle g_{\mathrm{SD}}(\hat{v})\,|\,T\rangle$, the asymptotic behaviour will be computed and then an approximate form will be computed. In all the following examples the statistics will be assumed to be stationary.

Case I: $T \ll \tau_{\mathrm{c}}$

In this case we can perform the following approximations:

$$\rho\int_{-T}^{0}\langle\alpha(t')\,|\,\hat{v}\rangle\,\mathrm{d}t' = \rho\int_{0}^{T}\langle\alpha(t')\,|\,\hat{v}\rangle\,\mathrm{d}t' \approx \rho\alpha(\hat{v})\,T. \tag{28}$$

Expanding the normalization factor in terms of $\sigma_\alpha/\langle\alpha\rangle$ and keeping terms up to second order, we get $(\sigma_\alpha/\langle\alpha\rangle \ll 2)$

$$\langle g_{\mathrm{SD}}(\hat{v})\,|\,T\rangle \approx \frac{\alpha(\hat{v})\left(1+\dfrac{T}{\tau_{\mathrm{M}}}\right)p(\hat{v})}{\langle\alpha\rangle\left(1+\dfrac{T}{\tau_{\mathrm{M}}}\dfrac{\alpha(\hat{v})}{\langle\alpha\rangle}\right)\left(1-\dfrac{\sigma_\alpha^2\,T/\tau_{\mathrm{M}}}{\langle\alpha\rangle^2(1+T/\tau_{\mathrm{M}})^2}\right)}, \tag{29}$$

where $\tau_{\mathrm{M}} = 1/\rho\langle\alpha\rangle$ as before. Two asymptotes are possible for this case.

(i) $T \gg \tau_{\mathrm{M}}$; the particle rate is high compared with the dead time T (saturated):

$$\langle g_{\mathrm{SD}}(\hat{v})\,|\,T\rangle \approx p(\hat{v}). \tag{30}$$

No bias is present!

(ii) $T \ll \tau_M$; low particle rate:

$$\langle g_{SD}(\hat{v})\,|\,T\rangle \approx \frac{\alpha(\hat{v})}{\langle\alpha\rangle}p(\hat{v}). \tag{31}$$

The statistics are those of the individual realization system.

Case II: $T \gg \tau_c$

In this case

$$\rho\int_{-T}^{0}\langle\alpha(t')\,|\,\hat{v}\rangle\,dt' = \rho\int_{0}^{T}\langle\alpha(t')\,|\,\hat{v}\rangle\,dt' \approx \rho\langle\alpha\rangle T. \tag{32}$$

For $T \ll \tau_M$ or $T \gg \tau_M$ we get

$$\langle g_{SD}(\hat{v})\,|\,T\rangle \approx \frac{\alpha(v)}{\langle\alpha\rangle}p(\hat{v}). \tag{33}$$

The statistics are those of the individual realization system independent of τ_M.

A better ideal of the behaviour of the statistics of the saturable detector can be gained by computing an approximate form. Here we will again assume Gaussian statistics, i.e.

$$\rho\int_{0}^{T}\langle\alpha(t')\,|\,v\rangle\,dt' = \rho\langle\alpha\rangle T + \rho(\alpha-\langle\alpha\rangle)\int_{0}^{T}R(t')\,dt', \tag{34}$$

where again $R(t)$ is the velocity autocorrelation function. The normalization is computed to second order in $\sigma_\alpha^2/\langle\alpha\rangle^2$:

$$\langle g_{SD}(\hat{v})\,|\,T\rangle \approx \frac{\alpha(\hat{v})\,(1+T/\tau_M)\,p(\hat{v})}{\langle\alpha\rangle\left(1+\dfrac{T-R_T}{\tau_M}+\rho\alpha R_T\right)\left(1-\dfrac{\sigma_\alpha^2}{\langle\alpha\rangle^2}\dfrac{R_T(\tau_M+T-R_T)}{(\tau_M+T)^2}\right)}, \tag{35}$$

where

$$R_T = \int_0^T R(t)\,dt'.$$

The normal situation is one where $T \ll \tau_c$. Under these conditions $R_T = T$. Thus normally (29) should be a useful approximation.

If only one component of velocity is measured, the statistics are calculated as before by integrating over the non-measured coordinates (see (22)). When this is done, one gets the same form as (29) (or (35)) except that $\hat{v}_x$ is substituted for $\hat{v}$ and $\rho\alpha(\hat{v})$ is interpreted as the measurement rate for $\hat{v}_x$.

In many systems it is reasonable to assume that $\alpha(v_x)$ is proportional to v_x, i.e.

$$\alpha(v_x) = |Av_x|. \tag{36}$$

Using (36) and (29), an approximation to the *measured* mean velocity may be computed if v_x is strictly positive. Again keeping terms to second order in $\sigma_{v_x}/\langle v_x\rangle$, one gets

$$\langle\hat{v}_x\rangle \approx \langle v_x\rangle\left(1+\frac{\sigma_{v_x}^2}{\langle v_x\rangle^2}\frac{1}{1+T/T_M}\right), \tag{37}$$

where $T_M = 1/\rho\langle|Av_x|\rangle$. This equation predicts the smooth transition from a biased estimate to an unbiased estimate of $\langle\hat{v}_x\rangle$ as the ratio of the dead time to the mean time between particles increases. Recall this formula was derived assuming that the dead time is small compared to the flow correlation time.

In a similar fashion, an expression for the turbulence intensity may be computed.

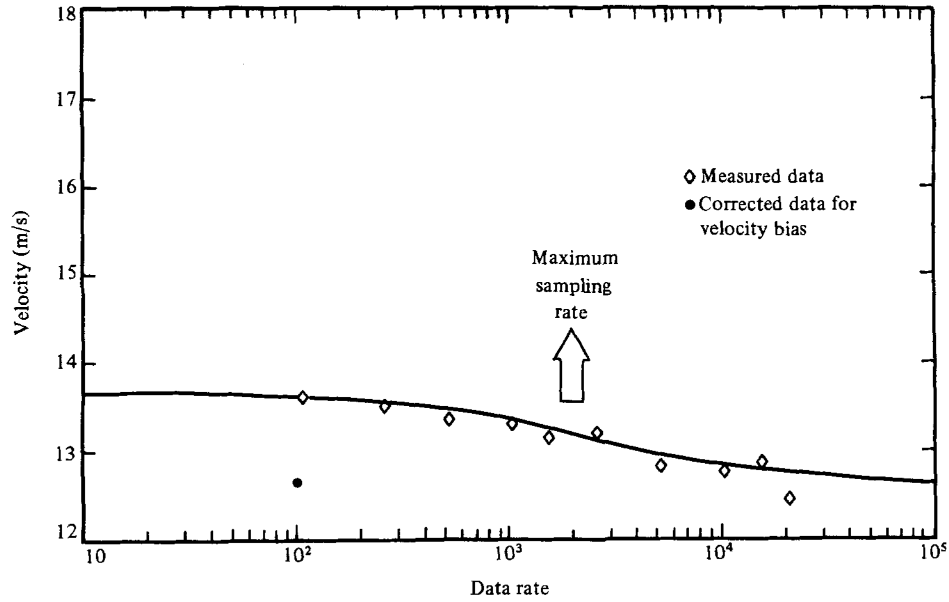

FIGURE 1. Velocity *vs.* data rate.

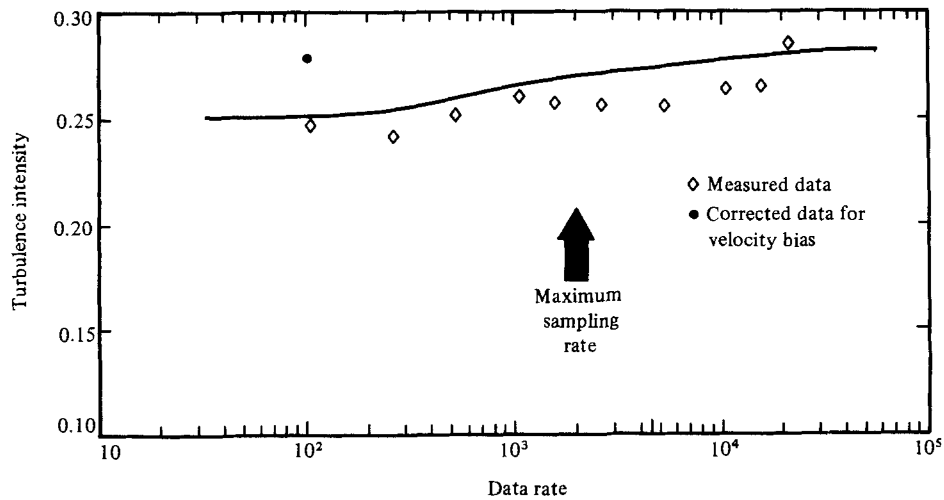

FIGURE 2. Turbulence intensity *vs.* data rate.

Again keeping terms only to sècond order in $\sigma_{v_x}/\langle v_x\rangle$,

$$\frac{\hat{\sigma}^2_{v_x}}{\langle \hat{v}_x\rangle^2} \approx \frac{\sigma^2_{v_x}}{\langle v_x\rangle^2} \frac{1-\dfrac{\sigma^2_{v_x}}{\langle v_x\rangle^2}\dfrac{(1-T/T_{\mathrm{M}})}{(1+T/T_M)^2}}{\left(1+\dfrac{\sigma^2_{v_x}}{\langle v_x\rangle^2}\dfrac{1}{(1+T/T_{\mathrm{M}})}\right)^2}. \tag{38}$$

The parameter $\hat{\sigma}^2_{v_x}$ is the measured variance of the flow. Again there is a smooth transition from a biased to an unbiased estimate as T/T_{M} goes to infinity.

Figures 1 and 2 show data taken by Stevenson *et al.* (1980), where the flow was kept constant while the seeding density was changed. The maximum data rate that the system can record is denoted by the arrow. This value is taken as $1/T$. Note that, as predicted, the bias decreases as the particle rate passes through the maximum recording rate. The solid line in figure 1 is equation (37) plotted using the ratio of the data rate to the maximum rate for T/T_M. The values of $\sigma_{v_x}^2/\langle v_x \rangle^2$ were estimated from the apparent asymptotes on the figure. Although (37) deviates from the data at some points, the overall agreement is quite gratifying.

Figure 2 is a plot of the measured turbulence intensity as a function of the data rate. The solid line is equation (38). Although the fit is not perfect, the trends are correct – the apparent turbulence intensity increases to an apparent asymptote as the data rate exceeds the maximum sampling rate.

3. Discussion

The calculations were performed for the full velocity vector to emphasize the three-dimensional nature of the problem. For instance, if one is only measuring a single component of velocity, its measurement rate may be strongly influenced by the other velocity components. This is especially true if the component measured is transverse to the main flow direction. Under these circumstances, the measurement rate need not be proportional to the measured velocity component.

The quantity $\rho\alpha(\hat{v})$ (or $\rho\alpha(v_x)$) can be interpreted as the mean *measurement* rate corresponding to the velocity $\hat{v}$. In §2 it was referred to as the mean particle *arrival* rate corresponding to the velocity $\hat{v}$. The mean measurement rate for velocity $\hat{v}$ can be influenced by non-uniform seeding as well as the other factors referred to above. A close examination will reveal that the derivations were performed assuming only that the number of events in the output of the initial processor in a small time interval $\Delta t (\ll \tau_c)$ is a random variable conditioned only by the velocity. This can be true even if there is a correlation between the particle density and the velocity or if the initial processor randomly misses some particles.

If the measurement rate fulfils the above assumption, the formulae derived above can be rewritten by deleting ρ and substituting the measurement rate $r(\hat{v})$ for $\alpha(\hat{v})$. The variable $N(\tau)$ is to be interpreted as the mean number of measurements in time τ. The variables $r(v)$ (or $r(v_x)$) and $N(\tau)$ are measurable (Durão, Laker and Whitelaw 1980), and thus the experimentalists can determine if his measurements fulfil the conditions of this work. Further, the parameter τ_c is the correlation time for $r(v)$, which is also measurable.

We know of one situation where the assumption of the randomness of the initial processor output fails. It is possible to arrange the geometry of a fringe system so that some counters will give multiple outputs for a *single particle*. This is especially true if an optical frequency shifter is used. Under these circumstances the initial processor output can occur in clusters with a well-defined spacing between events in the cluster, determined by the counter reset time and the particle velocity. The number of counts in some time intervals is no longer a random variable. The cluster can persist for the length of time the particle remains in the fringe region. One can easily observe this behaviour if a histogram is made of the measurement interarrival times. Cluster events will show up as a sharp peak, with an interarrival time approximately the inverse of the counter reset time.

The duration of the cluster is usually small compared with the flow correlation time. In this case, the output of the sample-and-hold processor will not be influenced by the cluster event since the velocity will not change significantly during its duration. Similarly, if a dead time T that is longer than the cluster duration is introduced, the output will be insensitive to the clusters.

Factors other than those mentioned here may affect the measurement rate as a function of velocity. Therefore we recommend that experimentalists measure the rate versus velocity. Failure to do this may result in incorrect assumptions about the measurement statistics.

The phrase 'individual-realization' statistics is now understood to mean that the output probability density is weighted by the measurement rate. The bias in the measured mean velocity, predicted by McLaughlin & Teidermann (1973), is the square of the turbulence intensity. For a system with 10% turbulence, this only amounts to a 1% change. Obviously, a measurement that attempts to measure the bias ought to be at least that accurate.

Without considering the bias, there are two major contributions to the estimate of a mean velocity in a fluctuating flow:

(i) the error per measurement (denoted by σ_M);

(ii) the fact that the flow fluctuates.

Since the variance of the estimate of the mean decreases as the number of *independent* samples of the mean velocity, the total measurement span compared with the flow correlation time is important. Measurements taken within the correlation time are not independent estimates of the mean. An estimate of the measurement variance of the mean, σ^2_{Mv}, can be derived (Saleh 1978), viz

$$\sigma^2_{Mv} \approx \frac{2\tau_c}{T_v}\left(\frac{\sigma^2_M T_M}{2\tau_c} + \sigma^2_{vx}\left(1+\frac{T_M}{2\tau_c}\right)\right), \tag{39}$$

where τ_c is the flow correlation time, T_v is the measurement interval, σ^2_{vx} is the variance of the velocity and T_M is the inverse of the particle rate, as before. The measurement error σ_M is assumed to be independent of the velocity fluctuation.

Usually σ_M is very small and can be neglected. The number of correlation times measured can easily be the main source of uncertainty in the estimate of the mean. Again using the example of 10% turbulence level, the measurement needs to extend over at least 200 correlation times. To get unequivocal evidence of bias, the error ought to be much smaller than the expected bias, so that 10000 correlation times may be a more realistic criterion.

The turbulence intensity levels in the work reported by Stevenson *et al.* (1980) were very high – sometimes around 30%. The analysis was performed by taking a fixed number of samples while varying the particle density. Thus, as the particle rate increases, the measurement interval decreases. Therefore the squared error in the estimate of the mean should increase approximately as the inverse of the stored particle rate. Since a counter typically gives errors of less than 0.5%, and their data indicates errors on the order of 3–5%, the cause could well be that not enough independent samples of the mean were taken. In fact, we know of no paper in the literature on bias errors where the effect of flow correlation time on the measurement accuracy was even mentioned.

The squared error in the estimate of the velocity variance decreases even more slowly as a function of the number of correlation times measured. It can be shown to decrease roughly as the square root of the number of correlation times. Thus the errors in the estimate of the turbulence intensity should be larger than those in the estimate of the mean. This is readily apparent in the data shown.

4. Conclusions

For both kinds of systems, sample-and-hold and saturable, the output statistics are a function of the particle arrival rate and the flow correlation time.

The statistics of the sample-and-hold processor are controlled by the dimensionless parameter τ_M/τ_c, the ratio of the mean measurement interarrival time to the flow correlation time. If this parameter is greater than 10, the output velocity statistics are the same as the flow statistics at the measurement region. If the parameter is less than 0.1, the output statistics are those of the individual realization case.

The statistics of the saturable system are controlled by τ_M/τ_c and T/τ_c, the ratio of the dead time to the flow correlation time. The asymptotes are

(i) $T/\tau_c \gg 1$, the output statistics remain those of the individual realization system no matter what the value of τ_M/τ_c;

(ii) $T/\tau_c \ll 1$, the output statistics are those of the flow if $\tau_M/\tau_c < 0.1$, and individual realization statistics if $\tau_M/\tau_c > 10$.

Since there has been little rigorous discussion of any of these parameters in the literature, it is not surprising that confusing experimental results have been attained.

The results derived here show that conditions are attainable where the output statistics are essentially identical to those of the flow. Aside from reducing the post-detection computation required, this result may be useful in flows where the particle density and velocity are correlated. This may occur near the edge of a jet where the seeding is not the same in the entrained fluid as in the jet core. If the particle rate and detector parameters are correctly set, the effect of non-uniform seeding may be eliminated.

The statistics of velocity detection in rotating machinery is different from that discussed in this paper. We hope to present results for those systems in a subsequent paper.

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Appendix

Let $\{t_i\}$ be a set of N times and let $t_j > t_i$ if $j > i$. Further let $x(t_i)$ be a random variable of zero mean. The set of random variables $x(t_i)$ is said to be Gaussian if their joint probability density is given by an equation of the form

$$p(\{x(t_i)\}) = (2\pi)^{\frac{1}{2}N}|\boldsymbol{\mu}|^{-\frac{1}{2}}\exp\left(-\tfrac{1}{2}\{x(t_i)\}'\,\boldsymbol{\mu}^{-1}\{x(t_j)\}\right), \tag{A 1}$$

where $\{x(t_i)\}$ is the column matrix of variables $x(t_i)$ and $\{x(t_i)\}'$ is its transpose, and $\boldsymbol{\mu}$ is the matrix whose elements are given by $\mu_{ij} = \langle x(t_i)\,x(t_j)\rangle$ (Saleh 1978). The matrix $\boldsymbol{\mu}^{-1}$ is its inverse and $|\boldsymbol{\mu}|$ is its determinant. If further the system is stationary,

$$\mu_{ij} = \langle x(t_i)\,x(t_j)\rangle = \langle x^2\rangle\, R(t_j - t_i), \tag{A 2}$$

where $R(\;\;)$ is the normalized autocorrelation function of $x(t)$. Without loss of generality, we can write

$$\langle x(t_i)\, x(t_j) \,|\, x(t_i)\rangle = (x(t_i))^2\, R(t_j - t_i). \tag{A 3}$$

Let $\alpha - \langle\alpha\rangle = x$. From the above we get

$$\langle \alpha(t)\,|\,v\rangle = \alpha R(t) + \langle\alpha\rangle\,(1 - R(t)). \tag{A 4}$$

Thus

$$\langle N(\tau)\,|\,v\rangle = \rho\alpha \int_0^\tau R(t)\,\mathrm{d}t + \rho\langle\alpha\rangle \int_0^\tau (1 - R(t))\,\mathrm{d}t. \tag{A 5}$$

Further,

$$\begin{aligned}\sigma_N^2 &= \langle (N(\tau) - \langle N(\tau)\,|\,v\rangle)^2\rangle \\ &= \rho^2(\alpha - \langle\alpha\rangle)^2 \left[\int_0^\tau (\tau - t)\, R(t)\,\mathrm{d}t - \left(\int_0^\tau R(t)\,\mathrm{d}t\right)^2\right].\end{aligned} \tag{A 6}$$

The form of the autocorrelation is arbitrary so long as it is a symmetric function of time and is 1 at $\tau = 0$. Here we choose

$$R(\tau) = \exp\left[-\frac{\tau^2}{2\tau_0^2}\right]. \tag{A 7}$$

With this definition, we get

$$\tau_c = \surd 2\left(\frac{\mathrm{d}^2R}{\mathrm{d}\tau^2}\bigg|_{\tau=0}\right)^{-\frac{1}{2}} = \surd 2\,\tau_0 \quad \text{(the microscale)},$$

$$\tau_E = \int_0^\infty R(t)\,\mathrm{d}t = (\tfrac{1}{2}\pi)^{\frac{1}{2}}\,\tau_0 \quad \text{(the macroscale)}.$$

Using $\tau_0 = \surd\frac{1}{2}\,\tau_c$, we can now construct the function $p(N\,|\,\tau, v)$:

$$p(N\,|\,\tau, v) = \left[\exp\left(-\frac{(N - \langle N\,|\,v\,\rangle)^2}{2\sigma_N^2}\right)\right]\Big/\int_0^\infty \exp\left(\frac{(N - \langle N\,|\,v\rangle)^2}{2\sigma_N^2}\right)\mathrm{d}N. \tag{A 8}$$

The integral

$$\int_0^\infty \langle \mathrm{e}^{-N(\tau)}\,|\,v\rangle\,\mathrm{d}\tau$$

can now be computed numerically after some algebraic manipulation:

$$\int_0^\infty \langle \mathrm{e}^{-N(\tau)}\,|\,v\rangle\,\mathrm{d}\tau = \int_0^\infty \frac{\left\{1 + \mathrm{erf}\left[\surd\frac{1}{2}\left(\frac{\langle N(\tau\,|\,v\rangle}{\sigma_N} - \sigma_N\right)\right]\right\}\exp\left[-(\langle N(\tau)\,|\,v\rangle - \frac{1}{2}\sigma_N^2)\right]}{1 + \mathrm{erf}\left[\langle N(\tau) v\rangle\, \surd 2\,\sigma_N\right]}\,\mathrm{d}\tau. \tag{A 9}$$

Very small time steps must be taken for the small values of τ because of the error function terms. Otherwise, the numerical integration is straightforward. Typical results of the integration (after multiplying by $\rho\alpha$) are given in table 1.

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